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THE EVASION PROBLEM IN NONSTATIONARY DIFFERENTIAL GAMES

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We examine the evasion problem for a nonstationary conflict-controlled system from contact with a given target set from any point not belonging to this set, over an extensive arbitrarily long time. We have obtained sufficient conditions for the evasion. The paper abuts the investigations in /1–9/.

1. Statement of the problem. The game equation is:

$$z^{\cdot} = f(t, z, u, v), \quad z \in E^n, \quad t \in [0, \infty) \tag{1.1}$$

Here $f(t, z, u, v)$ is a function continuous in the arguments and continuously differentiable in z, u and v are the control parameters of players P and E, chosen from convex compacta $U(t)$ and $V(t)$, depending continuously on t, and belonging for all t to some compacta U and V from the Euclidean space E^n. A terminal set M, which is a subset, is singled out in E^n. Player P strives to lead the trajectory of system (1.1) onto set M; player E's purpose is to prevent this contact. It is assumed that a constant C exists such that

$$(z, f(t, z, u, v)) \leqslant C(1 + \|z\|^2), \quad \forall t, z, \ u \in U(t), \ v \in V(t)$$

and the set $t, z, v \in V(t)$ is convex for any $f(t, z, U(t), v)$. The adversaries use ε-strategies (see /10/). The pair (t, z) is called a position.

We say that player E's ε-strategy (Γ_E) is given if for each position (t, z) there have been defined a number $\varepsilon(t, z)$, $\varepsilon(t, z) > 0$, and a function $\Gamma_E(\theta; t, z)$, $t \leqslant \theta \leqslant t + \varepsilon(t, z)$ satisfying the conditions: $v(\theta) = \Gamma_E(\theta; t, z)$ is a measurable function of θ, taking values in set $V(\theta)$. We say that player P's ε-startegy (Γ_P) is given if for each position (t, z) there has been defined a function $\Gamma_P(\theta; \varepsilon, v(\cdot), t, z)$ which associates with the position (t, z) the number $\varepsilon > 0$ and the function $v(\theta)$, $t \leqslant \theta \leqslant t + \varepsilon$, a function $u(\theta) = \Gamma_P(\theta; \varepsilon, v(\cdot), t, z)$ measurable for $t \leqslant \theta \leqslant t + \varepsilon$ and taking values in set $U(\theta)$. A uniquely defined trajectory of system (1.1), continuable onto the semi-infinite time interval $[t_0, \infty)$, corresponds to the collection $(t_0, z^\circ, \Gamma_P, \Gamma_E)$ /10/.

Evasion is possible in game (1.1) if a strategy Γ_E exists such that for any strategy Γ_P the corresponding trajectory of system (1.1), starting from any position (t_0, z°), $0 \leqslant t_0 < \infty$, $z^\circ = z(t_0) \overline{\in} M$, does not go onto set M for any value whatsoever of time t, $t \geqslant t_0$. We go on to state the results.

By L we denote the orthogonal complement of M in E^n and we assume that $\dim L = \nu \geqslant 2$. If R is some subspace of L, than by π_R we denote the operator of orthogonal projection from E^n onto R, while

$$\Psi_R = \{\psi: \psi \in R, \|\psi\| = 1\}$$

Without loss of generality we can accept that set M was singled out by relations depending only on the first m components of vector z, $m \leqslant n$. We denote $z_* = (z_1, \ldots, z_m)$, $f_* = (f_1, \ldots, f_m)$. Then the membership $z \in M$ is completely determined by vector z_*. We compute formally the time derivatives of z_* relative to system (1.1) with fixed controls

$$\frac{d^{i+1} z_*}{dt^{i+1}} = f_*^{(i)}(t, z, u, v) = \sum_{j=1}^{n} \frac{\partial f_*^{(i-1)}(\cdot)}{\partial z_j} f_j(\cdot) + \frac{\partial f_*^{(i-1)}(\cdot)}{\partial t}, \quad i = 1, 2, \ldots$$

$$f_*^{(0)}(\cdot) = f_*(\cdot)$$

2. Nonlinear game. General case. The following evasion theorem holds.

Theorem 1. Let there exist a number k, $k \leqslant \nu - 1$, a subspace R, $R \subset L$, $\dim R \geqslant k + 1$, and a function $l(t, z)$, continuous in the arguments, with values in R, such that:

a) the function $f(t, z, u, v)$ is continuous in all the arguments, the function $f_*(t, z, u, v)$ is continuously differentiable in (t, z) up to order $k - 1$, inclusive, while the functions $f_j(t, z, u, v)$, $j = \overline{m+1, n}$ are continuously differentiable in (t, z) up to order $k - s - 2$, where $s = s(j)$, $s \leqslant k - 1$ is such a number that $f_*^{(i)}(t, z, u, v)$, $i = \overline{0, s-1}$, do not depend upon z_j, but $f_*^{(s)}(t, z, u, v)$ now does depend on them;

b) the sets $\pi_R f_*^{(i)}(t, z, U(t), V(t))$, $i = \overline{0, k-2}$, consists of one point for any t, z, $t \geqslant 0$;

c) $\min\limits_{\psi \in \Psi_R} \max\limits_{v \in V(t)} \min\limits_{u \in U(t)} (\psi, f_*^{(k-1)}(t, z, u, v) - l(t, z)) > 0$

for all t, z, $t \geqslant 0$, $z \in M$.

Then evasion is possible in game (1.1).

Proof. We surround the subspace M by a shell

$$S(t) = \{z: \min_{\psi \in \Psi_R} \max_{v \in V(t)} \min_{u \in U(t)} (\psi, f_*^{(k-1)}(t, z, u, v) - l(t, z)) > 0\}$$

and we consider a position (t_0, z°) such that $z^\circ \in S(t_0) \setminus M$. We fix an element v_0 from $V(t_0)$, satisfying the condition

$$\max_{v \in V(t_0)} \min_{u \in U(t_0)} (\psi_0, f_*^{(k-1)}(t_0, z^\circ, u, v)) = \min_{u \in U(t_0)} (\psi_0, f_*^{(k-1)}(t_0, z^\circ, u, v_0)) \tag{2.1}$$

where the vector ψ_0 belongs to Ψ_R and satisfies the system of linear inequalities

$$(\psi_0, z^\circ) \geqslant 0, \quad (\psi_0, l(t_0, z^\circ)) \geqslant 0$$
$$(\psi_0, f_*^{(i)}(t_0, z^\circ, u, v)) \geqslant 0, \quad i = \overline{0, k-2} \tag{2.2}$$

System (2.2) is solvable for ψ_0 since condition (b) holds and $k + 1 \leqslant \nu$.

By $\Omega_r(z)$ we denote a sphere in E^n of radius r with center at point z. By virtue of condition (c) and of the assumptions on the game parameters, we can choose $\Omega_{r_0}(z^\circ)$ and the time interval $[t_0, t_0 + \delta_1]$, $\delta_1 > 0$, so small that the inequality

$$\min_{u \in U(t)} (\psi_0, f_*^{(k-1)}(t, z, u, v_0) - l(t_0, z^\circ)) \geqslant 0 \tag{2.3}$$

is fulfilled by continuity for $z \in \Omega_{r_0}(z^\circ)$ and $t \in [t_0, t_0 + \delta_1]$. From the assumptions on sets $U(t)$ and $V(t)$ and on the function $f(t, z, u, v)$, using the Gronwall lemma /11/ we get that $\delta_2 > 0$ exists such that a trajectory of system (1.1), starting from point $z^\circ = z(t_0)$ with a measurable control $u(t)$, $u(t) \in U(t)$, and with $v(t) = v_0$ does not leave $\Omega_{r_0}(z^\circ)$ during time δ_2. We denote $\tau_0 = \min(\delta_1, \delta_2)$ and we construct the evasion strategy Γ_E^*. To do this we set $\varepsilon(t_0, z^\circ) = \tau_0$, $v(t) = v_0$, $t_0 \leqslant t \leqslant t_0 + \tau_0$. Then control $u(t)$ is determined by strategy Γ_P and system (1.1) can be integrated on the interval $[t_0, t_0 + \tau_0]$ to obtain trajectory $z(t)$.

Let the position (t_0, z°) be such that $z^\circ \overline{\in} S(t_0)$. We select $\Omega_{r_0}(z^\circ)$ from the condition $\Omega_{r_0}(z^\circ) \cap M = \phi$. Then $\tau_0 > 0$ exists such that a trajectory of system (1.1), starting at point $z^\circ = z(t_0)$ with measurable controls $u(t)$ and $v(t)$, $u(t) \in U(t)$, $v(t) \in V(t)$, does not leave $\Omega_{r_0}(z^\circ)$ during time τ_0. Having set $\varepsilon(t_0, z^\circ) = \tau_0$ and having chosen some measurable control $v(t)$, $v(t) \in V(t)$, $t_0 \leqslant t \leqslant t_0 + \tau_0$, we construct the evasion strategy Γ_E^*. Control $u(t)$ is determined in accordance with Γ_P and, having integrated system (1.1) on $[t_0, t_0 + \tau_0]$, we obtain trajectory $z(t)$.

Let us consider the projection of the first m components of trajectory $z(t)$, $t_0 \leqslant t \leqslant t_0 + \tau_0$, $z^\circ \in S(t_0) \setminus M$, corresponding to the strategy pair (Γ_P, Γ_E^*), onto the direction of ψ_0. According to Taylor's theorem, by virtue of condition (a)

$$(\psi_0, z_*(t)) = (\psi_0, z_*^\circ) + \sum_{i=1}^{k-1} \frac{(t-t_0)^i}{i!} (\psi_0, f_*^{(i-1)}(t_0, z^\circ, u, v)) + R_k \tag{2.4}$$

$$R_k = \int_{t_0}^{t} \frac{(t-t_0-\zeta)^{k-1}}{(k-1)!} (\psi_0, f_*^{(k-1)}(\zeta, z(\zeta), u(\zeta), v_0))\, d\zeta =$$

$$\int_{t_0}^{t} \frac{(t-t_0-\zeta)^{k-1}}{(k-1)!} (\psi_0, f_*^{(k-1)}(\zeta, z(\zeta), u(\zeta), v_0) - l(t_0, z^\circ))\, d\zeta +$$

$$\frac{(t-t_0)^k}{k!} (\psi_0, l(t_0, z^\circ))$$

With due regard to relations (2.1)–(2.3), from formula (2.4) it follows that

$$(\psi_0, z_*(t)) > 0 \quad \text{for} \quad t_0 < t \leqslant t_0 + \tau_0 \tag{2.5}$$

By $\bar{S}(t)$ we denote a compactum, depending continuously on t, such that $\bar{S}(t) \subset S(t)$ for any t, $t \geqslant 0$. Let us show that on the set

$$Z' = \{\zeta, \bar{S}(\zeta)\}_{\zeta \in [t_0', T]} \tag{2.6}$$

we can choose $\varepsilon(t, z) \geqslant \tau > 0$, where τ depends only on t_0', T and $\bar{S}(t)$. From condition (c) follows

$$\min_{t \in [t_0', T]} \min_{z \in \bar{S}(t)} \min_{\psi \in \Psi_R} \max_{v \in V(t)} \min_{u \in U(t)} (\psi, f_*^{(k-1)}(t, z, u, v) - l(t, z)) = \Delta > 0 \tag{2.7}$$

Then, because the functions from (2.7) are continuous in all their arguments for the given Δ, there exist $\delta > 0$ and $r > 0$ such that when $t \in [t_0, t_0 + \delta]$, $z \in \Omega_r(z^\circ)$ the function

$$\min_{u \in U(t)} (\psi_0, f_*^{(k-1)}(t, z, u, v_0) - l(t_0, z^\circ))$$

remains nonnegative for any position $(t_0, z^\circ) \in Z'$. The trajectory $z(t)$, being an absolutely continuous function, satisfies a Lipschitz condition with constant K in the set $\bigcup_{\zeta\in[t_0',T]} \bar{S}(\zeta)$ and, consequently, the function $z(t)$, $z(t_0) = z^\circ$ does not leave the set $\Omega_r(z^\circ)$ during the time $t^\circ = r/K$. Thus, for any position $(t, z) \in Z'$ we can choose

$$\varepsilon(t, z) \geqslant \min(\delta, t^\circ) = \tau > 0$$

We showed earlier that for any position (t_0, z°), $z^\circ \in S(t_0) \setminus M$, a vector ψ_0, exists, $\psi_0 \in \Psi_R$, such that on some time interval $[t_0, t_0 + \tau_0]$ the projection of trajectory $z_*(t)$ onto the direction of ψ_0 increases monotonically, remaining positive, and, consequently, $z(t)$ cannot intersect set M on the interval $[t_0, t_0 + \tau_0]$. From the construction of the strategy $\Gamma_E{}^*$ for the position (t_0, z°), $z^\circ \overline{\in} S(t_0)$, it follows as well that $z(t)$ does not intersect M on some interval $[t_0, t_0 + \tau_0]$.

We now assume that the trajectory $z(t)$ starting from point $z^\circ = z(t_0) \overline{\in} M$, first intersects set M at some finite instant t_1, $t_1 > t_0$, i.e. $(\psi, z_*(t_1)) \leqslant 0$ for all $\psi \in L$. Then, by virtue of the assumptions on the parameters of game (1.1), $z(t)$, $t_0 \leqslant \bar{t} \leqslant t \leqslant t_1$ belongs to some compact set $\bar{S}(t)$ such that $\bar{S}(t) \subset S(t)$ for all t, $t \geqslant 0$. By what we proved earlier, we can choose $\varepsilon(t, z) \geqslant \tau > 0$ on the set $\{\zeta, \bar{S}(\zeta)_{\zeta\in[\bar{t},t_1]}\}$. Let $t_1 = t_* + \beta$, where $\beta \leqslant \varepsilon(t_*, z(t_*))$, $t_* > \bar{t}$, t_* is the instant at which control $v(t)$ is presented. Then $\varepsilon(t_*, z(t_*)) \geqslant \tau$ and according to (2.5) there exists a vector $\psi_* \in \Psi_R$, $R \subset L$, such that

$$(\psi_*, z_*(t)) > 0 \quad \text{for} \quad t_* < t \leqslant t_* + \varepsilon(t_*, z(t_*))$$

which leads to a contradiction. By the same token we have proved Theorem 1.

3. Nonlinear game with separated nonlinear controlling part. We assume that the right-hand side in the equation of motion (1.1) is of the form

$$f(t, z, u, v) = g(t, z) + h(t, u, v)$$

The following theorem is valid.

Theorem 2. Let the functions $g(t, z)$ and $h(t, u, v)$ be continuous in all their arguments and let there exist a two-dimensional subspace R, $R \subset L$, and a continuous function $l(t)$ with values in R, such that

$$\min_{\psi\in\Psi_R} \max_{v\in V(t)} \min_{u\in U(t)} (\psi, h(t,u,v) - l(t)) > 0, \qquad t \in [0, \infty)$$

Then evasion is possible in game (1.1).

Proof. The reasoning is analogous to that in the proof of Theorem 1 with the sole difference that $E^n \setminus M$ is chosen as $S(t)$ and that for the position (t_0, z°), $z^\circ \overline{\in} M$, the element v_0 from $V(t_0)$ is chosen from the relation

$$\max_{v\in V(t_0)} \min_{u\in U(t_0)} (\psi_0, h(t_0, u, v)) = \min_{u\in U(t_0)} (\psi_0, h(t_0, u, v_0))$$

moreover, $\psi_0 \in \Psi_R$, $(\psi_0, z^\circ) \geqslant 0$, $(\psi_0, g(t_0, z^\circ) + l(t_0)) \geqslant 0$. Inequality (2.3) is replaced by

$$(\psi_0, g(t, z)) + \min_{u\in U(t)} (\psi_0, h(t, u, v_0)) \geqslant 0$$

The set Z' acquires the form $[t_0', T] \times Z$, where Z is a compactum from E^n, while relation (2.7) is changed to

$$\min_{t\in[t_0',T]}\min_{\psi\in\Psi_R}\max_{v\in V(t)}\min_{u\in U(t)}(\psi, h(t,u,v)-l(t))=\Delta>0$$

4. Linear game with nonlinear controlling part. Let the right-hand side in the equation of motion (1.1) have the form

$$f(t, z, u, v) = A(t) z + h(t, u, v)$$

where $A(t)$ is a square ($n \times n$)-matrix. By $a_i(t)$ we denote the i-th row of matrix $A(t)$, while by $A_*(t)$, the matrix consisting of the first m rows of matrix $A(t)$. We formally form a sequence of matrices with the aid of the recurrence relation

$$B_i(t) = B_{i-1}(t) A(t) + B_{i-1}^{\cdot}(t), \quad i=1, 2, \ldots$$

where $B_0(t) = A_*(t)$, while the symbol $B^{\cdot}(t)$ denotes the derivative of matrix $B(t)$. Then the following theorem is valid.

Theorem 3. Let there exist a number k, $k \leqslant \nu - 1$, a subspace R, $R \subset L$, $\dim R \geqslant k+1$, and a continuous function $l(t)$ with values in R, such that:

a) the function $h(t,u,v)$ is continuous in all arguments, the matrix $A_*(t)$ and the function $h_*(t,u,v)$ are continuously differentiable in t up to order $k-1$, inclusive, the functions $a_j(t)$, $h_j(t,u,v)$, $j=\overline{m+1, n}$, are continuously differentiable in t up to order $k-s-2$, where $s=s(j)\leqslant k-1$ is a number such that $B_i(t)z$, $i=\overline{0, s-1}$, do not depend upon z_j, but $B_s(t)z$ now does depend upon them;

b) the sets $\pi_R B_i(t) h(t, U(t), V(t))$, $i=\overline{0, k-3}$, consist of one point for $t\in[0, \infty)$;

c) $$\min_{\psi\in\Psi_R}\max_{v\in V(t)}\min_{u\in U(t)}(\psi, B_{k-2}(t)h(t,u,v)-l(t))>0$$

for all $t\in[0, \infty)$.

Then evasion is possible in game (1.1).

Proof. We consider a position (t_0, z°), $z^\circ \overline{\in} M$. We fix an element v_0 from $V(t_0)$, satisfying the condition

$$\max_{v\in V(t_0)}\min_{u\in U(t_0)}(\psi_0, B_{k-2}(t_0)h(t_0,u,v))=\min_{u\in U(t_0)}(\psi_0, B_{k-2}(t_0)h(t_0,u,v_0)) \quad (4.1)$$

moreover, ψ_0 belongs to Ψ_R and satisfies the system of linear inequalities

$$\begin{aligned}
&(\psi_0, z^\circ)\geqslant 0 && (4.2)\\
&(\psi_0, B_i(t_0)z^\circ + h_*^{(i)}(t_0,u,v))\geqslant 0, \quad i=\overline{0,k-2}\\
&\left(\psi_0, B_{k-1}(t_0)z^\circ + \left.\frac{\partial h_*^{(k-2)}(t,u,v)}{\partial t}\right|_{t=t_0} + l(t_0)\right)\geqslant 0\\
&h_*^{(i)}(t,u,v) = B_{i-1}(t)h(t,u,v) + \frac{\partial h_*^{(i-1)}(t,u,v)}{\partial t}, \quad i=\overline{1,k-1}\\
&h_*^{(0)}(t,u,v) = h_*(t,u,v)
\end{aligned}$$

By virtue of condition (b), inequalities (4.2) do not depend on u and v. From relations (4.1), (4.2), and condition (c) we obtain

$$(\psi_0, B_{k-1}(t_0)z^\circ) + \min_{u\in U(t_0)}(\psi_0, h_*^{(k-1)}(t_0,u,v_0)) =$$

$$\left(\psi_0, B_{k-1}(t_0) z^\circ + \left.\frac{\partial h_*^{(k-2)}(t, u, v)}{\partial t}\right|_{t=t_0} + l(t_0)\right) + \min_{u \in U(t_0)} (\psi_0, B_{k-2}(t_0) h(t_0, u, v_0) - l(t_0)) > 0$$

Consequently, there exists $\Omega_{r_0}(z^\circ)$ and an interval $[t_0, t_0 + \delta_1]$, $\delta_1 > 0$, so small that

$$(\psi_0, B_{k-1}(t) z) + \min_{u \in U(t)} (\psi_0, h_*^{(k-1)}(t, u, v_0)) \geqslant 0$$

From the assumptions on the parameters of game (1.1) follows the existence of $\delta_2 > 0$ such that a trajectory of system (1.1), starting at point $z^\circ = z(t_0)$ with a measurable control $u(t)$, $u(t) \in U$, and with $v(t) = v_0$, does not leave $\Omega_{r_0}(z^\circ)$ during time δ_2. Having set $\tau_0 = \min(\delta_1, \delta_2)$, $\varepsilon(t_0, z^\circ) = \tau_0$ and $v(t) = v_0$, $t_0 \leqslant t \leqslant t_0 + \tau_0$, we construct strategy $\Gamma_E{}^*$. The control $u(t)$ is determined in accord with strategy Γ_P and system (1.1) can be integrated on the interval $[t_0, t_0 + \tau_0]$ to obtain trajectory $z(t)$.

With due regard to relations (4.1), (4.2) and condition (c), from Taylor's formula we obtain an estimate for the projection of trajectory $z_*(t)$ corresponding to the collection $(t_0, z^\circ, \Gamma_P, \Gamma_E^*)$ onto the direction of ψ_0

$$(\psi_0, z_*(t)) > 0, \quad t_0 < t \leqslant t_0 + \tau_0$$

Let us show that on the set $[t_0', T] \times Z$, where Z is a compactum from E^n, we can choose $\varepsilon(t, z) \geqslant \tau > 0$, where τ depends only on this set.

By virtue of condition (c)

$$\min_{t \in [t_0', T]} \min_{\psi \in \Psi_R} \max_{v \in V(t)} \min_{u \in U(t)} (\psi, B_{k-2}(t) h(t, u, v) - l(t)) = \Delta > 0$$

Then $\delta > 0$ and $r > 0$ exist such that the function

$$(\psi_0, B_{k-1}(t) z) + \min_{u \in U(t)} (\psi_0, h_*^{(k-1)}(t, u, v_0))$$

remains nonnegative for $t \in [t_0, t_0 + \delta]$, $z \in \Omega_r(z^\circ)$, where (t_0, z°) is an arbitrary position from $[t_0', T] \times Z$. In Z the trajectory $z(t)$ satisfies a Lipschitz condition with constant K and, consequently, does not leave $\Omega_r(z^\circ)$ during the time $t^\circ = \delta / K$. Thus, for any position $(t, z) \in [t_0', T] \times Z$ we can choose

$$\varepsilon(t, z) \geqslant \min(\delta, t^\circ) = \tau > 0$$

The subsequent reasonings are analogous to those in the concluding part of the proof of Theorem 1.

5. Linear game with linear controlling part. We assume that the right-hand side in the equation of motion (1.1) has the form

$$f(t, z, u, v) = A(t) z - u + v$$

By $W_X(\psi)$ we denote the support function of a convex closed set X and by $B^*(t)$, the matrix adjoint to $B(t)$. For a comparison with the results in /8/ we present the following corollary stemming from Theorem 3.

Corollary. Let there exist a number k, $k \leqslant \nu - 1$, a subspace R, $R \subset L$, $\dim R \geqslant k + 1$, and a continuous function $l(t)$ with values in R, such that:

a) the matrix $A_*(t)$ is continuously differentiable up to order $k-1$, inclusive, while the functions $a_j(t)$, $j=\overline{m+1,\ n}$, are continuously differentiable up to order $k-s-2$, where $s=s(j)\leqslant k-1$ is a number such that $B_i(t)z$, $i=\overline{0,\ s-1}$, do not depend upon z_j, but $B_s(t)z$ now does depend upon them;

b) the sets $\pi_R B_i(t)(-U(t)+V(t))$, $i=\overline{0,\ k-3}$ consist of one point for $t\in[0,\ \infty)$;

c) $\min\limits_{\psi\in\Psi_R}[W_{V(t)}(B_{k-2}^*(t)\psi)-W_{U(t)}(B_{k-2}^*(t)\psi)-(\psi,l(t))]>0$

for all $t\in[0,\ \infty)$.

Then evasion is possible in game (1.1).

6. Example. The laws of motion of the pursuer and evader are given, respectively, by the equations

$$x^{(p)}+C_1(t)x^{(p-1)}+\ldots+C_{p-1}(t)x^{\cdot}+C_p(t)x=u$$

$$y^{(q)}+D_1(t)y^{(q-1)}+\ldots+D_{q-1}(t)y^{\cdot}+D_q(t)y=v$$

Here x, y are vectors in a Euclidean space E^n, $n\geqslant 2$, $x^{(i)}$, $y^{(i)}$ are the i th-order time derivatives of x, y; $C_i(t)$, $i=\overline{1,p}$, $D_j(t)$, $j=\overline{1,q}$ are matrices depending continuously on t, $t\in[0,\infty]$, $u\in\bar U(t)$, $v\in\bar V(t)$, where $\bar U(t)$, $\bar V(t)$ are convex compacta from E^n, depending continuously on t and belonging to some compacta U, V for each t. The terminal set is $M=\{(x,y): x=y\}$. It is assumed that $q\leqslant n-1$ and $\dim\bar V(t)=n$, $t\in[0,\infty)$. Having set

$$z_1=x,\ z_2=x^{\cdot},\ \ldots,\ z_p=x^{(p-1)},z_{p+1}=y,z_{p+2}=y^{\cdot},\ \ldots,z_{p+q}=y^{(q-1)}$$

we pass to the equivalent system of differential equations.

In the case being examined M is given by the equation

$$z_1=z_{p+1},\ L=\{\underbrace{\alpha,\ 0,\ldots,0}_{p},\ -\alpha,\ 0,\ \ldots,\ 0\}$$

where α is an arbitrary vector of E^n. We denote

$$U(t)=\{(\underbrace{0,\ldots,0}_{p-1},-u,0,\ldots,0):u\in\bar U(t)\}$$

$$V(t)=\{(\underbrace{0,\ldots,0}_{p+q-1},v):v\in\bar V(t)\}$$

where 0 is the null-vector of the space E^nq. It is not difficult to verify that when $q\leqslant p$

$$\pi_L B_i(t)(-U(t)+V(t))=(0,\ \ldots,\ 0),\ i=\overline{0,\ q-2}$$

and, hence, $\pi_R B_i(t)(-U(t)+V(t))=(0,\ \ldots,0)$ for any $R\subset L$. According to the Corollary it is sufficient to fulfill one of the following conditions for evasion to be possible: (1) $q<p$; (2) when $q=p$ there exist a subspace R, $R\subset L$, $\dim R\geqslant q+1$, and a continuous function $l(t)$ with values in R, such that

$$\min_{\psi\in\Psi_R}[W_{\bar V(t)}(\psi)-W_{\bar U(t)}(\psi)-(\psi,l(t))]>0,\qquad \forall t\in[0,\infty)$$

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APPLICATION OF THE PERTURBATION METHOD TO SOME OPTIMAL CONTROL PROBLEMS

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We examine a quasilinear optimal control system. We justify the applicability of the perturbation method to some control problems. Various systems for constructing an approximate solution of control problems with a small parameter were presented in /1 – 4/. A number of practical optimal control problems can be described by systems containing linear terms and small, in general, nonlinear perturbing factors. The scheme of successive approximations of the perturbation method developed below, can prove to be useful for the analytic investigations of such problems. The method is justified for quasilinear systems with a quadratic performance index.

1. Let a control system be given by the equation